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On the residues and special values of Poincaré Series

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ABSTRACT

Let G be the universal cover of $\mathrm{SL}(2,\mathbf{R})$, Γ a lattice in G and χ a unitary character of Γ. In this paper we use residues and special values of the Poincaré series $\mathbf{M}(\nu,\chi)$, for the construction of (χ,Γ)-square integrable automorphic forms on G. In particular, we show that for low weight $1 < r \leq 2$, the special values at $\nu = r-1$ obtained by meromorphic continuation of $\mathbf{M}^{\eta}(\nu,\chi)$, as η varies, generate the space of holomorphic cusp forms of weight r and multiplier system v_χ. We also prove a completeness result for holomorphic forms of weight $0 < r \leq 1$ by using residues of the family.

1. INTRODUCTION

Let G be the universal cover of $\mathrm{SL}(2,\mathbf{R})$, Γ a lattice in G containing the center of G, and χ a unitary character of Γ. The purpose of the present paper is to apply the methods in [MW] to the construction of (χ,Γ)-square integrable automorphic forms on G. We shall use a holomorphic family $\mathbf{M}(\nu,\chi)$ of functions, defined by means of a matrix entry of a principal series representation, which transform according to a character η of the unipotent subgroup N. This family, originally defined in a half-plane admits a meromorphic continuation to $\mathbf{C}$. The main goal of this paper is to prove completeness results in $L^2_d(\Gamma\backslash G,\chi)$ for the residues and special values of $\mathbf{M}(\nu,\chi)$. We will show that the residues of $\mathbf{M}(\nu,\chi)$ in $\{\nu : \mathrm{Re}(\nu) \geq 0\}$ generate, as η varies, all $(\mathfrak{g},K)$-modules of square

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integrable (χ, Γ)-automorphic forms, except those corresponding to the discrete series representations D_r, D^-_{-r}, $r > 1$ (see Theorem 3.3). As an application, we get a completeness result for holomorphic (or antiholomorphic) forms of weight $0 < r \leq 1$ (see Theorem 4.1 (ii)). Furthermore, we will show that for weights r with $1 < r \leq 2$, the analytic continuation of $\mathbf{M}(\nu, g, \phi_r, \chi)$ is holomorphic at $\nu = r - 1$ and (for $\eta = \eta_\lambda$ with $\lambda > 0$) it defines a holomorphic cusp form. These special values generate the space of holomorphic cusp forms of weight r and multiplier system v_χ (c.f. Theorem 4.1).

The family $\mathbf{M}(\nu, \chi)$ has been previously considered in a classical setting in [Ne], [Ni] (for $r = 0$, $\chi = 1$) and [He], and also in [MW] ($\chi = 1$) and [Br2]. Some of the results on the meromorphic continuation of the family can be found in the references above, mainly in [Br2]. On the other hand the goals and methods in [Br2] are different and in particular L^2-completeness theorems are not discussed, thus our results give new information in this direction. The authors wish to thank R. Bruggeman for very useful comments on an earlier version of this paper, which have led to clarifications and some simplifications.

2. PRELIMINARIES

2.1. Spectral decomposition of $L^2(\Gamma\backslash G, \chi)$

Let G be the universal cover of $G_0 = \mathrm{SL}(2, \mathbf{R})$ and $\pi : G \to G_0$ the canonical projection. G is generated by elements $\{n(x), a(y), k(\theta) \mid x, \theta \in \mathbf{R}, y > 0\}$ such that

$$\pi(n(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \quad \pi(a(y)) = \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y^{-1}} \end{bmatrix} \quad \pi(k(\theta)) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

If we set $p(x + iy) = n(x)a(y)$, $\pi(g) := \bar{g}$ and $\bar{g} \cdot z$ denotes the standard action of $\bar{g}$ on $z \in H$, the Poincaré upper half plane, we have the relations

$$\begin{aligned} &n(x_1)n(x_2) = n(x_1 + x_2), \quad a(y_1)a(y_2) = a(y_1 y_2) \quad x_1, x_2 \in \mathbf{R}, \ y_1, y_2 \in \mathbf{R}^+, \\ &k(\theta_1)k(\theta_2) = k(\theta_1 + \theta_2), \quad a(y)n(x) = n(yx)a(y) \quad \theta_1, \theta_2 \in \mathbf{R}, \ x \in \mathbf{R}, \ y \in \mathbf{R}^+ \\ &k(\theta)p(z) = p(\pi(k(\theta)) \cdot z)k(\theta - \arg|_{(-\pi,\pi]} e^{i\theta}(-z\sin\theta + \cos\theta)). \end{aligned}$$

Set $N = \{n(x) \mid x \in \mathbf{R}\}$, $A = \{a(y) \mid y > 0\}$, $K = \{k(\theta) \mid \theta \in \mathbf{R}\}$, $M = \{k(m\pi) \mid m \in \mathbf{Z}\}$, the center of G, and $P = NAM$. Any element $g \in G$ decomposes uniquely $g = nak$, with $n \in N$, $a \in A$ and $k \in K$. We use invariant measures dn, da, dk on N, A and K corresponding respectively to dx, dy/y, $d\theta/\pi$. On G we will normalize the Haar measure so that $\int_G f(g)dg = \int_N \int_A \int_K f(nak)a^{-2\rho} dn\, da\, dk$ for $f \in C_c(G)$, with $a(y)^\rho = \sqrt{y}$. Let $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{a}$, $\mathfrak{k}$ denote the Lie algebras of G, N, A, K respectively. If $\mathfrak{g}$ is identified with the Lie algebra of G_0 and we set

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad W = X - Y,$$

then X, H, W span respectively $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$. We denote $E^{\pm} = H \pm i(X+Y) \in \mathfrak{g}_c = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ and $C = (H^2/4) - (H/2) + XY \in U(\mathfrak{g})$, the Casimir element of G. Moreover, we will identify $\mathfrak{a}_c^*$ with $\mathbf{C}$ via the map $\nu \mapsto \nu(H)$.

From now on we will fix Γ, a noncocompact lattice in G and χ, a character of Γ. We will assume, for simplicity, that Γ contains M. We note that any lattice Γ on G can be enlarged to the lattice $\tilde{\Gamma} = \Gamma M$, and the character χ can be extended to a unitary character $\tilde{\chi}$ on $\tilde{\Gamma}$. In general, any lattice in G intersects M nontrivially and pushes down to a lattice in G_0 (see for instance [Ho], Proposition 9).

If Q is a parabolic subgroup of G, then $Q = kPk^{-1}$ for some $k \in K$, that is $Q = N_Q A_Q M$, with $N_Q = kNk^{-1}$, $A_Q = kAk^{-1}$. Q is said to be Γ-cuspidal if $\Gamma_{N_Q} := \Gamma \cap N_Q \neq 1$. Γ acts by conjugation on the set of such subgroups and an orbit of this action is called a cusp of Γ. As is well known, this set is finite. A cuspidal parabolic subgroup Q will be said to be *regular* with respect to χ (resp. *irregular*) if $\chi|_{\Gamma_{N_Q}} \equiv 1$ (resp. $\chi|_{\Gamma_{N_Q}} \not\equiv 1$). We shall fix a complete set of representatives, $P_1, \ldots, P_r$ of Γ-cuspidal parabolic subgroups of G, so that P_j is regular for $1 \leq j \leq s$, and irregular for $s+1 \leq j \leq r$. We let $P_1 = P$ and fix $k_j \in K$ so that $P_j = k_j P k_j^{-1}$, for $1 \leq j \leq r$.

If $Q = kPk^{-1}$, set $\mathfrak{a}_Q = Ad(k)\mathfrak{a}$ and if $\nu \in \mathfrak{a}_c^*$, $H \in \mathfrak{a}_Q$, set $k\nu(H) = \nu(Ad(k^{-1})H)$. Let ω be an open relatively compact subset of N_Q such that $\pi : \omega \to \Gamma_{N_Q} \backslash N_Q$ is surjective. If $t > 0$, let $A_{Q,t}^+ = \{a \in A_Q \mid a^{k\rho} > e^{t/2}\}$. A set of the form $\mathcal{S}_{Q,\omega,t} = \omega \times A_{Q,t}^+ \times K$ is called a Q-Siegel set for Γ. By a well known result of reduction theory, if P_j, ω_j are as above and $t_j > 0$, for $j = 1 \ldots r$, then $\Gamma \backslash G - \bigcup_{j=1}^r \pi(\mathcal{S}_{P_j,\omega_j,t_j})$ is compact.

The characters of M are of the form $\xi(k(n\pi)) = e^{in\pi\tau}$, for some $\tau = \tau_\xi$, $-1 < \tau \leq 1$. If $\chi \in \hat{\Gamma}$ we shall also write $\tau_\chi = \tau_{\chi|_M}$.

If $r \in \mathbf{R}$, set $\phi_r(k(\theta)) = e^{ir\theta}$. A smooth function f on G is (χ, Γ)-automorphic of weight r if, for any $k(\theta) \in K$, $g \in G$, $\gamma \in \Gamma$ it verifies

$$f(\gamma g k(\theta)) = \chi(\gamma) f(g) \phi_r(k(\theta)),$$

and if, furthermore, f is a finite linear combination of eigenfunctions of C and satisfies a growth condition. Namely, for each $Q = kPk^{-1}$ a regular parabolic subgroup and $X \in \mathfrak{g}_c$, there exist $C_X > 0$ and $d \in \mathbf{R}$ such that $|Xf(kg)| \leq C_X a(g)^{d\rho}$. If Q is a regular cusp, the *Q-constant term* of f is defined by

$$(1) \qquad f_Q(g) = \int_{\Gamma_{N_Q} \backslash N_Q} f(ng)\, dn.$$

If $f_Q \equiv 0$ for any Q, f is said to be a cusp form.

We shall denote by $\mathcal{A}(\Gamma \backslash G, \chi)$ (resp. $\mathcal{A}_0(\Gamma \backslash G, \chi)$) the space of (χ, Γ)-automorphic forms (resp. automorphic cusp forms). We note that $\mathcal{A}(\Gamma \backslash G, \chi)$ is generated by functions of weight r with $r \equiv \tau_\chi\,(2)$. The (χ, Γ)-automorphic forms correspond to classical Γ-automorphic forms on H of weight r and multiplier system $v = v_\chi$ (see [Br, §4.4]). Here v_χ on $\bar{\Gamma}$ is given by $v_\chi(\bar{\gamma}) = \chi(\sigma(\gamma))$ for $\gamma \in \Gamma$, where $\sigma : G_0 \to G$, is the section such that

$$\sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = p\left(\frac{ai+b}{ci+d}\right)k(arg_{|_{(-\pi,\pi]}}(-ci+d)).$$

We now define the Eisenstein series. Let Q be a regular cuspidal parabolic subgroup of G. Let $\phi \in \hat{K}$ be such that $\phi_{|M} = \chi_{|M}$. If $g \in G$, Re $\nu > 1$, set

$$(2) \qquad E(Q,\nu,g,\phi,\chi) = \sum_{\gamma \in \Gamma_Q \backslash \Gamma} a_Q(\gamma g)^{\nu+\rho}\phi(k_Q(\gamma g))\chi(\gamma)^{-1}.$$

This series converges uniformly on compacta for $\mathrm{Re}\,\nu > 1$ and lies in $\mathcal{A}(\Gamma\backslash G,\chi)$. Furthermore $E(Q,\nu,g,\phi,\chi)$ admits a meromorphic continuation to $\mathbf{C}$ which is holomorphic for Re $\nu = 0$ (see [He] or [Br], for instance).

Given V a Banach representation of G, let V_∞, V_ω and V_K denote, respectively, the space of C^∞ vectors, analytic vectors and C^∞, K-finite vectors in V. Also, if $\tau \in \hat{K}$, let $V[\tau]$ and $V_K[\tau]$ be the τ-isotypic components of V and V_K respectively. We shall make use of some facts on the spectral decomposition of the right regular representation of G of $L^2(\Gamma\backslash G,\chi)$ (see [Ro], or [Ho]).

Theorem 2.1. *As a representation of G, $L^2(\Gamma\backslash G,\chi) = L^2_d(\Gamma\backslash G,\chi) \oplus L^2_c(\Gamma\backslash G,\chi)$, where $L^2_d(\Gamma\backslash G,\chi)$ (resp. $L^2_c(\Gamma\backslash G,\chi)$) decomposes discretely (resp. continuously). If $r \equiv \tau_\chi$ (2), there exist $\{\psi_l\}_{l=1}^\infty$, a complete orthonormal system in $L^2_d(\Gamma\backslash G,\chi)_K[\phi_r]$, so that every ψ_l generates an irreducible $(\mathfrak{g},K)$-submodule. Let $\{P_1,\cdots,P_s\}$ be a complete system of representatives of regular cuspidal parabolic subgroups. If $f \in L^2(\Gamma\backslash G,\chi)[\phi_r]$, then, in the L^2-sense,*

$$(3) \qquad f = \sum_{l=1}^{\infty} \langle f,\psi_l\rangle\psi_l + \sum_{j=1}^{s} \frac{1}{2\pi i} \int_{\mathrm{Re}\,\mu=0} f_j(\mu)E(P_j,\mu,\cdot,L(k_j)\phi_r,\chi)\,d\mu$$

with $f_j \in L^2(i\mathbf{R},d\mu)$, for $1 \le j \le s$.

2.2. Representation theory of G

We keep the notation from previous subsections. If $\xi \in \hat{M}$, $\nu \in \mathfrak{a}_c^*$, let $H^{\xi,\nu}$ denote the principal series representation of G (see [Br, §3]). Set $s = k(\pi/2)$ and let as above $\tau = \tau_\xi \in (-1,1]$ be such that $\xi(k(\pi)) = e^{i\pi\tau}$. When restricted to K, $H^{\xi,\nu}$ decomposes $H^{\xi,\nu} = \sum_{r\equiv\tau_\xi(2)} \mathbf{C}\phi_r$. Furthermore, the action of $E^\pm$ on $H_K^{\xi,\nu}$ is given by

$$(4) \qquad E^\pm \cdot \phi_r = (1+\nu\pm r)\phi_{r\pm 2}.$$

We now recall some facts on the Kunze–Stein intertwining operator (see [GW]).

(i) For any $f \in H_\infty^{\xi,\nu}$, the integral $A(\xi,\nu)f(k) = \int_N \pi_{\xi,\nu}(snk)f(1)dn$ converges absolutely and uniformly on compacta for $k \in K$ and $\mathrm{Re}\,\nu > 0$.

(ii) The map $\nu \to \Gamma(\nu)^{-1}A(\xi,\nu)$ has a holomorphic continuation to $\mathbf{C}$ with values on the continuous intertwining operators on $H_\infty^{\xi,\nu}$.

(iii) If $r \equiv \tau_\xi$ (2) then $A(\xi,\nu)\phi_r = c_r(\nu)\phi_r$ with

$$c_r(\nu) = \frac{2^{1-\nu}\pi e^{ir\pi/2}\Gamma(\nu)}{\Gamma\left(\frac{1+\nu+r}{2}\right)\Gamma\left(\frac{1+\nu-r}{2}\right)}.$$

In particular, (iii) implies that $A(\xi,-\nu)\cdot A(\xi,\nu) = \mu_\xi(\nu)^{-1}\,\mathrm{Id}$, where $\mu_\xi(\nu)^{-1}$ equals

$$\frac{-4\pi\cos\frac{\pi(\nu+r)}{2}\cos\frac{\pi(\nu-r)}{2}}{\nu\sin(\pi\nu)}.$$

We note that this expression does not depend on r but only on ξ. Also, if $H_K^{\xi,\nu}$ pushes down to a representation of $\mathrm{SL}(2,\mathbf{R})$ (i.e. if $\tau = 0$ or $\tau = 1$) then $\mu_\xi(it) = t/2\pi\tanh(\pi t/2)$ and $\mu_\xi(it) = t/2\pi\coth(\pi t/2)$, respectively.

The action of $E^\pm$ (see (4)) implies that reducibility of $H_K^{\xi,\nu}$ occurs when $(1+\nu)\equiv \pm r\,(2)$. The following theorem gives the composition series of $H_K^{\xi,\nu}$ (see [Pu], [Br]).

Theorem 2.2. *Let ξ be a character of M, $\tau = \tau_\xi$, $\nu\in\mathbf{C}$.*

1. *If $\tau \neq 0, 1$, $1+\nu\equiv\pm\tau\,(2)$, $H_K^{\xi,\nu}$ has a unique $(\mathfrak{g},K)$-submodule $D_{1+\nu}^{\pm} = \sum_{\pm r\geq 1+\nu}\mathbf{C}\phi_r$ and $(H^{\xi,\nu}/D^{\pm}_{\pm(1+\nu)})\simeq D^{\mp}_{\pm(\nu-1)}$.*
2. *If $\tau = 0, 1$, $1+\nu\equiv\tau\,(2)$ and $\nu\geq 0$, $H_K^{\xi,\nu}$ contains two irreducible $(\mathfrak{g},K)$-submodules: $D^{\pm}_{\pm(1+\nu)}$ and furthermore $H^{\xi,\nu}/(D^+_{1+\nu}\oplus D^-_{-(1+\nu)})\simeq F_\nu$ where F_ν is an irreducible $(\mathfrak{g},K)$-module of dimension ν.*
3. *If $\tau = 0, 1$, $1+\nu\equiv\tau\,(2)$ and $\nu\leq -1$, $H^{\xi,\nu}$ contains a unique $(\mathfrak{g},K)$-submodule $F_{-\nu}$. It has dimension $-\nu$ and $H^{\xi,\nu}/F_{-\nu}\simeq D^+_{1-\nu}\oplus D^-_{\nu-1}$.*

Any irreducible $(\mathfrak{g},K)$-module V is isomorphic to a submodule of $H_K^{\xi,\nu}$, for some ξ,ν. Furthermore if V is unitarizable, then either $V\simeq H_K^{\xi,\nu}$, with $\nu\in i\mathbf{R}-\{0\}$, $V\simeq D^{\pm}_{\pm r}$, with $r>0$, or $V\simeq H_K^{\xi,\nu}$, with $\nu\in\mathbf{R}$, $0\leq|\nu|<1-|\tau_\xi|$.

2.3. Whittaker functionals on $H^{\xi,\nu}$

If we set for $f\in H_\infty^{\xi,\nu}$, $\delta(\xi,\nu)f = f(1)$, then $\delta(\xi,\nu)$ defines a continuous functional on $H_\infty^{\xi,\nu}$ and $X\delta(\xi,\nu) = 0$. Fix $\eta = \eta_\lambda\in\hat{N}$, $\lambda\in\mathbf{R}^*$, so that $\eta(n(x)) = e^{2\pi i\lambda x}$. If we apply the operator $T(-\nu)$ on $H_\infty^{\xi,\nu}$ given by the formal series

$$T(-\nu) = \sum_{k=0}^{\infty}\frac{(-1)^k(d\eta)^k(X)}{k!\,\Gamma(\nu+k+1)}\,Y^k,$$

then $w(\xi,\nu) = T(-\nu)\delta(\xi,\nu)$ defines a continuous Whittaker functional on $H_\omega^{\xi,\nu}$, i.e. it satisfies $Xw(\xi,\nu) = d\eta(X)w(\xi,\nu)$ (see [GW] Theorem 6.1, or [MW], Theorem A.1.8). The Jacquet Whittaker vector is defined, if $f\in H_\infty^{\xi,\nu}$, by

$$\mathcal{J}_{\xi,\nu}(f) := \int_N \eta(n)^{-1}\pi_{\xi,\nu}(sn)f(1)\,dn. \tag{5}$$

This integral is convergent for $\mathrm{Re}\,\nu>0$ and has an analytic continuation to a continuous functional on $H_\infty^{\xi,\nu}, \mathcal{J}(\nu)$ such that $\mathcal{J}(\nu)\cdot\pi_{\xi,\nu}(n) = \eta(n)\mathcal{J}(\nu)$, $n \in N$ ([GW] Theorem 7.6).

As a functional on $H_\omega^{\xi,\nu}$, $\mathcal{J}(\nu)$ can be expressed in terms of $w(\xi,\nu)$ and $w_s(\xi,\nu) := w(\xi,-\nu)\cdot A(\xi,\nu)$ and satisfies a functional equation. We have (see [MW, Lemma 1.3] and [GW, Lemma 7.5 (7.30), Theorem 7.6]):

$$\mathcal{J}(\nu) = a(\nu)w(\xi,\nu) + b(\nu)w_s(\xi,\nu) \tag{6}$$

$$\text{with } a(\nu) = -2\pi e^{i(\pi/2)r}(2\pi|\lambda|)^\nu \frac{\cos\dfrac{\pi(\nu+sg(\lambda)r)}{2}}{\sin(\pi\nu)}, \quad b(\nu) = \Gamma(-\nu+1) \tag{7}$$

$$\mathcal{J}(-\nu)\cdot A(\xi,\nu) = \gamma_\xi(\nu)\mathcal{J}(\nu) \tag{8}$$

$$\text{where } \gamma_\xi(\nu) = \frac{a(-\nu)}{\Gamma(-\nu+1)} = 2e^{i(\pi/2)r}(2\pi|\lambda|)^{-\nu}\Gamma(\nu)\cos\frac{\pi(\nu-sg(\lambda)r)}{2}. \tag{9}$$

We next discuss briefly the relationship between $w(\xi,\nu), \mathcal{J}_{\xi,\nu}$ and the Whittaker functions.
If $s \notin (-\frac{1}{2})\mathbf{N}$ set

$$M_{k,s}(y) = y^{s+1/2}e^{-y/2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2}+s-k)_n}{n!\,(1+2s)_n}y^n,$$

where $(a)_n = \prod_{j=0}^{n-1}(a+j)$ and $(a)_0 = 1$. This function is a solution of the Whittaker differential equation

$$F''(y) + \left(-\frac{1}{4}+\frac{k}{y}+\frac{\frac{1}{4}-s^2}{y^2}\right)F(y) = 0.$$

Furthermore, $M_{k,s}(y) \ll y^{1/2+\mathrm{Re}\,s}$, as $y \downarrow 0$, and has exponential growth as $y \to \infty$.
Another solution is given for generic s, by

$$W_{k,s}(y) = \frac{\Gamma(2s)}{\Gamma(\frac{1}{2}+s-k)}M_{k,-s}(y) + \frac{\Gamma(-2s)}{\Gamma(\frac{1}{2}-s-k)}M_{k,s}(y). \tag{10}$$

This function satisfies $W_{k,s}(y) \sim y^k e^{-(y/2)}$ as $y \to +\infty$. For special values of the parameters, $M_{k,s}$ and $W_{k,s}$ are a multiple of each other, a fact which will be useful to us. Indeed, assume $s-k+\frac{1}{2} = -h$, with $h \in \mathbf{Z}^{\geq 0}$ and $2s \notin \mathbf{Z}^{<0}$. In this case, from the expression of $M_{k,s}$ and the asymptotics of $W_{k,s}$ as $y \to +\infty$ one gets

$$W_{k,s}(y) = (-1)^h(2k-2h)_h M_{k,s}(y) = y^{s+(1/2)}e^{-(y/2)}p_h(y) \tag{11}$$

where $p_h(y)$ is a polynomial of degree h and $p_0(y) = 1$.

Proposition 2.3 *(see [GW, Introd.]). Let $\eta(n(x)) = e^{2\pi i\lambda x}$, with $\lambda \in \mathbf{R}^*$. If $z = 4\pi\lambda e^{2t}$ and $a_t = \exp(tH)$, $t \in \mathbf{R}$, then $F_1(z) = w(\xi,\nu)(\pi_{\xi,\nu}(a_t)\phi_r)$ and $F_2(z) = \mathcal{J}_{\xi,\nu}(\pi_{\xi,\nu}(a_t)\phi_r)$, are solutions of the Whittaker differential equation with $k = sg(\lambda)(r/2)$ and $s = \nu/2$. Furthermore*

$$F_1(z) = \frac{(4\pi|\lambda|)^{-(\nu+1)/2}}{\Gamma(\nu+1)} M_{((sg(\lambda)r)/2),\nu/2}(|z|),$$

$$F_2(z) = \frac{\pi e^{i(\pi/2)r}(\pi|\lambda|)^{(\nu-1)/2}}{\Gamma\left(\frac{1+\nu+sg(\lambda)r}{2}\right)} W_{((sg(\lambda)r)/2),\nu/2}(|z|).$$

Let $f \in \mathcal{A}(\Gamma\backslash G,\chi)$ $[\phi_r]$ generating an irreducible $(\mathfrak{g},K)$-module V_f. By Proposition 2.2, there exist $\xi \in \hat{M}$, $\mu \in \mathbf{C}$ such that V_f is isomorphic to a quotient of $H_K^{\xi,\mu}$. If $\eta \in (\Gamma_N\backslash N)\hat{}$, denote by $f_{P,\eta}$ the η-Fourier coefficient of f,

$$(12) \qquad f_{P,\eta}(g) = \int_{\Gamma_N\backslash N} \eta(n)^{-1} f(ng)\, dn.$$

Lemma 2.4. *Let $\eta = \eta_\lambda \in (\Gamma_N\backslash N)\hat{}$ and let $f \in \mathcal{A}(\Gamma\backslash G,\chi)$ $[\phi_r]$ be such that V_f is isomorphic to a subquotient of $H_K^{\xi,\mu}$. There exists $c(\eta,f) \in \mathbf{C}$ such that, if $a \in A$,*

$$(13) \qquad f_{P,\eta}(a) = c(\eta,f)\, W_{sg(\lambda)(r/2),(\mu/2)}(4\pi|\lambda|a^\alpha) = c_\mu(\eta,f)\,\mathcal{J}(\mu)(\pi_\mu(a)\phi_r).$$

Here,

$$c_\mu(\eta,f) = e^{-ir\pi/2}\pi^{-1}\Gamma\left(\frac{1+\mu+sg(\lambda)r}{2}\right)(\pi|\lambda|)^{(1-\mu)/2} c(\eta,f),$$

$$\text{if } \frac{1+\mu+sg(\lambda)r}{2} \notin \mathbf{Z}^{\le 0}.$$

Proof. The condition on V_f implies that $\mathcal{C} f_{P,\eta} = ((\mu^2-1)/4) f_{P,\eta}$. By a computation using that $\mathcal{C} = (H^2/4) - (H/2) + XY$, we get

$$XY f_{P,\eta}(a_t) = (X^2 - XW) f_{P,\eta}(a_t) = -((2\pi\lambda e^{2t})^2 + r(2\pi\lambda e^{2t})) f_{P,\eta}(a_t).$$

This implies that if $z = 4\pi\lambda e^{2t}$, then

$$\left[\frac{1}{4z^2}\frac{d^2}{dt^2} - \frac{1}{2z^2}\frac{d}{dt} - \left(\frac{1}{4} + \frac{r}{2z} + \frac{1-\mu^2}{4z^2}\right)\right] f_{P,\eta}(a_t) = 0$$

hence, if $\lambda > 0$, it follows that $F(z) = f_{P,\eta}(a_t)$ satisfies the Whittaker equation of parameters $k = r/2$ and $s = \mu/2$. Now, since f lies in $\mathcal{A}(\Gamma\backslash G,\chi)$, the growth condition implies that there exists $c(\eta,f) \in \mathbf{C}$ such that $f_{P,\eta}(a_t) = c(\eta,f)\, W_{(r/2),(\mu/2)}(z)$. If $\lambda < 0$ one puts $G(z) = F(-z)$ and argues as in the case $\lambda > 0$. The last equality in the lemma follows from the formula in [GW, Lemma 7.3 (III)]. □

Remark 2.5. If we let $f = E(P_j,\nu,L(k_j)\phi_r,\chi)$ then $c_\nu(\eta,f) = e^{-ir\pi} \times D_\eta^1(P_j,P,\bar{\nu},\chi)$, and $D_\eta^1(P_j,P,\bar{\nu},\chi)$ is given, for $\operatorname{Re}\nu > 1$ by an absolutely convergent Dirichlet series which can be computed by the method in [MW] 2.7.

3.1. The Poincaré series $\mathbf{M}(\nu)$

We keep the notation from previous subsections. Fix $\eta \in \hat{N}$, $\xi \in \hat{M}$, satisfying the compatibility conditions $\eta_{|\Gamma_N} = \chi_{|\Gamma_N}$, $\chi_{|M} = \xi$. Let $\lambda \in \mathbf{R}^*$ be so that $\eta = \eta_\lambda$, where $\eta_\lambda(n(x)) = e^{2\pi i \lambda x}$, for $n(x) \in N$. If $\nu \in \mathbf{C}$, $v \in H_K^{\xi,\nu}$ and $g \in G$ set

$$(14) \qquad M^\eta(\xi, \nu, g, v) = w(\xi, \nu)(\pi_{\xi,\nu}(g)v)$$

where $w(\xi, \nu)$ is as in §2.3. The properties of the Whittaker M-function, Proposition 2.3 and the fact that if $X \in U(\mathfrak{g})$, $XM(\xi, \nu, g, v) = M(\xi, \nu, g, Xv)$, imply that if ω is a compact subset of $\{\nu \mid \operatorname{Re}\nu > 1\}$, there exists $C_{\omega, X} > 0$ such that

$$(15) \qquad |XM(\xi, \nu, g, v)| \leq C_{\omega, X} a(g)^{\operatorname{Re}\nu + \rho}$$

for $\nu \in \omega$ and $g \in G$ with $a(g)^\rho < T$.
Thus, (15) and the convergence of the Eisenstein series imply that the series

$$(16) \qquad \mathbf{M}^\eta(\xi, \nu, g, v, \chi) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} \chi(\gamma^{-1}) M^\eta(\xi, \nu, \gamma g, v)$$

defines a C^∞ function in $\{\nu : \operatorname{Re}\nu > 1\} \times G$, holomorphic in ν (the conditions on η, χ and ξ, imply that the summed function is invariant under Γ_P). We now define two auxiliary functions which are useful in the study of $\mathbf{M}^\eta(\xi, \nu, g, v, \chi)$.

Let $\phi = \phi_T \in C^\infty(G)$ be left N-invariant, right K-invariant, non increasing on A with $\phi(a) = 0$ if $a^\rho \geq 2T$ and $\phi(a) = 1$ if $a^\rho \leq T$. If $\lambda_\nu = (\nu^2 - 1)/4$, set

$$(17) \qquad \tilde{\mathbf{M}}(\xi, \nu, g, v, \chi) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} \phi(\gamma g) M(\xi, \nu, \gamma g, v) \chi(\gamma^{-1})$$

$$(18) \qquad \check{\mathbf{M}}(\xi, \nu, g, v, \chi) = (\mathcal{C} - \lambda_\nu I) \tilde{\mathbf{M}}(\xi, \nu, g, v, \chi).$$

By the definition, $\mathbf{M}(\nu, \phi_r, \chi) - \tilde{\mathbf{M}}(\nu, \phi_r, \chi)$ is locally a finite sum of translates of $M(\nu, g)$. Hence $\tilde{\mathbf{M}}(\nu, \phi_r, \chi)$ is C^∞ and holomorphic in ν in any region $\mathbf{M}(\nu, \phi_r, \chi)$ is so.

Lemma 3.1 *(c.f.* [MW], 2.2, 2.3*). If* $\operatorname{Re}\nu > 1$ *and* $X \in U(\mathfrak{g})$ *then* $X\tilde{\mathbf{M}}(\nu, \phi_r, \chi)$ *is bounded in* $\Gamma \backslash G$ *and holomorphic in* ν. $\check{\mathbf{M}}(\nu, \phi_r, \chi)$ *can be analytically continued to a* C^∞ *function on* $\mathbf{C} \times \Gamma \backslash G$, *holomorphic in* ν *and with compact support in* $\Gamma \backslash G$.

We now introduce some notation. If $\lambda_\nu = (\nu^2 - 1)/4$, the eigenvalue of $\mathcal{C}$ in $H_\infty^{\xi,\nu}$, let $Q(\Gamma, \chi) = \{\nu \in \mathbf{C} : \lambda_\nu \neq 0$ is an eigenvalue of C in $L_d^2(\Gamma \backslash G, \chi)\}$. Thus $Q(\Gamma, \chi)$ is a closed discrete subset of $i\mathbf{R} \cup (-1, 1)$.

The next result summarizes the main properties of $\mathbf{M}(\nu, \chi)$ and its truncation $\tilde{\mathbf{M}}(\nu, \chi)$. Many of the assertions can be found in [MW] ($G = SL(2, \mathbf{R})$, $\chi = 1$) and [Br2] (see also [Ne], for $r = 0$, $\chi = 1$, and [He, §9.7]). A new addition is the expression of the residue at $\nu = 0$ in assertion (ii) in the case of simple pole.

This will allow a detailed study of the singularity of $\mathbf{M}(\nu,\chi)$ at $\nu=0$ (see Theorem 3.3 (ii), (iii)).

Theorem 3.2. *Let $\eta=\eta_\lambda\in\hat{N}$, $\xi\in\hat{M}$, with $\eta_{|\Gamma_N}=\chi_{|\Gamma_N}$, $\chi_{|M}=\xi_\tau$ and let $r\equiv\tau$,* mod 2. *We have*

(i) $\tilde{\mathbf{M}}(\nu,\phi_r,\chi)$ *and* $\mathbf{M}(\nu,\phi_r,\chi)$ *can be meromorphically continued to* $\mathbf{C}$ *and the principal parts at the poles coincide. If $\nu_0\neq 0$ is a pole with* $\operatorname{Re}\nu\geq 0$ *then ν_0 lies in $Q(\Gamma,\chi)$ and* $\operatorname{Res}_{\nu=\nu_0}\mathbf{M}(\nu,\phi_r,\chi)$ *is a square integrable automorphic form. Furthermore* $\tilde{\mathbf{M}}(\nu,\phi_r,\chi)$ *is of moderate growth and, if ν is not a pole, it lies in $L^2_\infty(\Gamma\backslash G,\chi)$ if* $\operatorname{Re}\nu>0$ *(resp. in $L^{2-\alpha}_\infty(\Gamma\backslash G,\chi)$ if* $\operatorname{Re}\nu\geq 0$*, for any α with $0<\alpha<2$).*

(ii) *The order of $\nu_0=0$ as a pole of $\mathbf{M}(\nu,\phi_r,\chi)$ is at most 2, and in this case,* $\lim_{\nu\to 0}\nu^2\mathbf{M}(\nu,\phi_r,\chi)$ *is a square integrable automorphic form. If $\nu=0$ is a simple pole, then* $\operatorname{Res}_{\nu=0}\mathbf{M}(\nu,\phi_r,\chi)=f_r+g_r$ *with f_r square-integrable and*

$$g_r=-e^{ir\pi/2}\cos\frac{\pi r}{2}\sum_{j=1}^{s}D^1_\eta(P_j,P,0,\chi)E(P_j,0,L(k_j)\phi_r,\chi)$$

where $D^1_\eta(P_j,P,\nu,\chi)$ is as in Remark 2.5.

(iii) $\mathbf{M}(\nu,\phi_r,\chi)$ *satisfies the functional equation:*

$$(19)\qquad \begin{cases} (4\pi|\lambda|)^{\nu/2}\nu\Gamma\left(\dfrac{1+\nu-sg(\lambda)r}{2}\right)\mathbf{M}(\nu,\phi_r,\chi) \\ \qquad -(4\pi|\lambda|)^{-(\nu/2)}\nu\Gamma\left(\dfrac{1-\nu-sg(\lambda)r}{2}\right)\mathbf{M}(-\nu,\phi_r,\chi) \\ \qquad =\dfrac{(\pi|\lambda|)^{-(\nu/2)}2\pi e^{ir\pi/2}}{\Gamma\left(\frac{1-\nu+sg(\lambda)r}{2}\right)}\displaystyle\sum_{j=1}^{s}D^1_\eta(P_j,P,-\nu,\chi)E(P_j,\nu,L(k_j)\phi_r,\chi). \end{cases}$$

Proof. Statements (i) and (iii) are proved by the arguments in [MW] (see Theorems 2.5, 2.6, 2.8). Thus, we will only discuss assertion (ii).

Firstly, the difference between $\mathbf{M}(\nu,\phi_r,\chi)$ and $\tilde{\mathbf{M}}(\nu,\phi_r,\chi)$ is locally a finite sum of holomorphic functions on $\mathbf{C}$, hence they have the same poles and the same principal parts at the poles. On the other hand, if $\tilde{\mathbf{M}}_d(\nu,\phi_r,\chi)$ and $\tilde{\mathbf{M}}_c(\nu,\phi_r,\chi)$ denote the discrete and continuous parts of $\tilde{\mathbf{M}}(\nu,\phi_r,\chi)$, as in [MW] one shows that $\tilde{\mathbf{M}}_d(\nu,\phi_r,\chi)$ has a pole of order $h\leq 2$ at $\nu=0$ and if $h=2$, $\lim_{\nu\to 0}\nu^2\mathbf{M}(\nu,\phi_r,\chi)$ is a square integrable automorphic form. Finally $\tilde{\mathbf{M}}_c(\nu,\phi_r,\chi)$ is holomorphic in the half-plane $\{\nu:\operatorname{Re}\nu\geq 0\}$, except possibly for a simple pole at $\nu=0$ with residue

$$\operatorname{Res}_{\nu=0}\tilde{\mathbf{M}}_c(\nu,\phi_r,\chi)=\sum_{j=1}^{s}d_j(0,0)E(P_j,0,L(k_j)\phi_r,\chi).$$

The remaining task will be to determine the functions $d_j(\nu,\pm\nu)$, $1\leq j\leq s$. This will lead to the expression in (ii) of the theorem. We will use the argument in the proof of Lemma 2.4 in [MW]. Since $\check{\mathbf{M}}(\nu,\phi_r,\chi)=(C-\lambda_\nu)\tilde{\mathbf{M}}(\nu,\phi_r,\chi)$ it follows that

$$(20)\qquad d_j(\nu,\mu)=\frac{\mu^2-\nu^2}{4}\langle\tilde{\mathbf{M}}(\nu,\phi_r,\chi),E(P_j,-k_j\bar{\mu},\phi_r,\chi)\rangle.$$

Now, the inner product in this expression equals, for $\operatorname{Re}\nu>1$,

$$(21)\qquad \begin{cases}\displaystyle\int_{\Gamma_P\backslash G}\phi(x)M(\nu,x,\phi_r)\overline{E(P_j,-k_j\bar{\mu},x,L(k_j)\phi_r,\chi)}\,dx\\ \displaystyle\qquad = D_\eta^1(P_j,P,-\mu,\chi)e^{ir\pi}\int_A a^{-2\rho}\phi(a)w(\nu)(\pi_\nu(a)\phi_r)\overline{\mathcal{J}(-\bar{\mu})(\pi_{-\bar{\mu}}(a)\phi_r)}\,da\end{cases}$$

by Lemma 2.4 and the remark after it. Now we use the expansion

$$w(\nu)(\pi_\nu(a)\phi_r)=\frac{a^{\nu+\rho}}{\Gamma(\nu+1)}\sum_{k=0}^{\infty}a_{k,r}(\nu)a^{k\alpha}$$

where the $a_{k,r}(\nu)$ are rational functions with poles in $-\mathbf{N}$ and $a_{0,r}\equiv 1$. This series converges absolutely and uniformly on $\Omega\times\operatorname{supp}\phi$, for any Ω a compact subset of $\mathbf{C}-\{-\mathbf{N}\}$. Substituting in (21) and using (7) we get

$$\begin{aligned}&\frac{D_\eta^1(P_j,P,-\mu,\chi)}{\Gamma(\nu+1)}\\ &\times\left\{\int_A\phi(a)a^{-2\rho}\frac{a(-\mu)}{\Gamma(-\mu+1)}\sum_{k,l=0}^{\infty}a_{k,r}(\nu)\overline{a_{l,r}(-\bar{\mu})}a^{\nu-\mu+2\rho+(k+l)\alpha}\,da\right.\\ &\qquad\left.+\int_A\phi(a)a^{-2\rho}\frac{b(-\mu)c_r(-\mu)}{\Gamma(\mu+1)}\sum_{k,l=0}^{\infty}a_{k,r}(\nu)\overline{a_{l,r}(\bar{\mu})}a^{\nu+\mu+2\rho+(k+l)\alpha}\,da\right\}.\end{aligned}$$

Interchanging summation and integration and using (9) we get that for any ν,μ with $\operatorname{Re}(\nu+\mu)>0$, $\operatorname{Re}(\nu-\mu)>0$, this equals

$$\begin{aligned}&\frac{D_\eta^1(P_j,P,-\mu,\chi)}{\Gamma(\nu+1)}\\ &\times\left\{\gamma_\xi(\mu)\sum_{k,l=0}^{\infty}a_{k,r}(\nu)\overline{a_{l,r}(-\bar{\mu})}\left(\frac{e^{T((\nu-\mu)+2(k+l))}}{(\nu-\mu)+2(k+l)}+F_{k,l}(\nu,-\mu)\right)\right.\\ &\qquad\left.+c_r(-\mu)\sum_{k,l=0}^{\infty}a_{k,r}(\nu)\overline{a_{l,r}(\bar{\mu})}\left(\frac{e^{T((\nu+\mu)+2(k+l))}}{(\nu+\mu)+2(k+l)}+F_{k,l}(\nu,\mu)\right)\right\}\end{aligned}$$

where $F_{k,l}(\nu,\mu)$ is holomorphic in $\mathbf{C}^2$.

Thus, if ν is not a pole of either one of $\gamma_\xi(\mu)$, $D_\eta^1(P_j,P,-\mu,\chi)$ or $c_r(-\mu)$, the only term which may contribute to a pole at $\mu=\nu$ is the one corresponding to $k=l=0$ in the first summand. Hence, by (20) and (9), we have

$$(22)\qquad d_j(\nu,\nu)=-D_\eta^1(P_j,P,-\nu,\chi)\,\frac{\nu}{2}\,\frac{\gamma_\xi(\nu)}{\Gamma(\nu+1)}$$

$$(23)\qquad =-D_\eta^1(P_j,P,-\nu,\chi)\,|2\pi\lambda|^{-\nu}e^{i\pi r/2}\cos\pi\left(\frac{\nu-sg(\lambda)r}{2}\right).$$

Arguing in the same way, if $\mu \to -\nu$ one needs only consider the term corresponding to $k = l = 0$ in the second sum. We thus obtain

$$d_j(\nu, -\nu) = -D_\eta^1(P_j, P, \nu, \chi)\, \frac{c_r(\nu)}{2\Gamma(\nu)}. \tag{24}$$

Substituting the expression of $c_r(\nu)$ from §2.2 (iii), yields the expression for g_r in (ii) of the theorem. □

3.2. An inner product formula

In this subsection we will prove an inner product formula which will imply a completeness result for the residues of the **M**-series in the halfplane $\{\nu : \operatorname{Re}\nu \geq 0\}$. We will also study in detail the singularity at $\nu = 0$ (compare with [Ne] and [MW], Theorem 3.2 in the case $\chi = 1$).

Theorem 3.3. *Let $f \in L_d^2(\Gamma\backslash G, \chi)_K[\phi_r]$ be such that V_f is isomorphic to an irreducible quotient of $H_K^{\xi,\bar{\mu}}$ with $\operatorname{Re}\mu \geq 0$. If $\eta = \eta_\lambda \in \hat{N}$, with $\eta_{|\Gamma_N} = \chi_{|\Gamma_N}$ we have*

(i) *If $\mu \neq 0$,*

$$\langle \operatorname{Res}_{\nu=\mu} \mathbf{M}^\eta(\nu, \phi_r, \chi), f\rangle = \frac{2(4\pi|\lambda|)^{(1-\mu)/2}}{\mu\Gamma\left(\dfrac{1+\mu-sg(\lambda)r}{2}\right)}\,\overline{c(\eta, f)} \tag{1}$$

$\mathbf{M}^\eta(\nu, \phi_r, \chi)$ has a simple pole at $\nu = \mu$ if and only if there exists f as in the statement with $c(\eta, f) \neq 0$.

(ii) *If $\mu = 0$, then if $c(\eta, f) \neq 0$ for some f as in the statement, $\mathbf{M}^\eta(\nu, \phi_r, \chi)$ has a simple pole at $\nu = 0$ if r is an odd integer and $r\lambda > 0$, and it has a double pole otherwise. If h is the order of the pole, then $\lim_{\nu\to 0} \nu^h \mathbf{M}^\eta(\nu, \phi_r, \chi)$ is a square integrable automorphic form and we have*

$$\langle \lim_{\nu\to 0} \nu^h \mathbf{M}^\eta(\nu, \phi_r, \chi), f\rangle = hu(r, \lambda)(4\pi|\lambda|)^{1/2}\,\overline{c(\eta, f)} \tag{2}$$

with $u(r, \lambda) = (-1)^{(|r|+1)/2}(1)_{(|r|-1)/2}$ if r is an odd integer with $r\lambda > 0$ (i.e. $h = 1$) and

$$u(r, \lambda) = \Gamma\left(\frac{1-sg(\lambda)r}{2}\right)^{-1},$$

otherwise, ($h = 2$).

(iii) *If $\mu = 0$ and if $c(\eta, f) = 0$ for any f as above, $\mathbf{M}^\eta(\nu, \phi_r, \chi)$ has at most a simple pole at $\nu = 0$ and*

$$\begin{cases} \operatorname{Res}_{\nu=0} \mathbf{M}^\eta(\nu, \phi_r, \chi) \\ \quad = \operatorname{Res}_{\nu=0} \tilde{\mathbf{M}}_c^\eta(\nu, \phi_r, \chi) \\ \quad = -e^{ir\pi/2} \cos\dfrac{\pi r}{2} \displaystyle\sum_{j=1}^{s} D_\eta^1(P_j, P, 0, \chi) E(P_j, 0, L(k_j)\phi_r, \chi). \end{cases} \tag{3}$$

In this case, there is a simple pole with residue as in (3), *or* $\mathbf{M}^\eta(\nu, g, \phi_r, \chi)$ *is holomorphic at* $\nu = 0$. *The last possibility occurs if the expression in* (3) *is zero, in particular, if r is an odd integer.*
If $\mathbf{M}^\eta(\nu, \phi_1, \chi)$ *is holomorphic at* $\nu = 0$, *we have by* (19):

$$\mathbf{M}^\eta(0, \phi_1, \chi) = \frac{\pi i}{2} \sum_{j=1}^{s} D_\eta^1(P_j, P, 0, \chi) E(P_j, 0, L(k_j)\phi_1, \chi).$$

Remark. In all cases, by Theorem 2.2, the assumptions on μ, r imply in each case that the inner product in (i) or (ii) is zero if and only if $c(\eta, f)$ is zero. It follows that any square integrable automorphic form $f \neq 0$, such that V_f is irreducible and isomorphic to a unitarizable principal series or to D_r^+, D_{-r}^- for $0 < r \leq 1$, can not be perpendicular to all residues of $M(\nu, \chi)$. Furthermore, in Theorem 4.1, we shall see that those $(\mathfrak{g}, K)$-modules of type D_r^+, D_{-r}^-, for $r > 1$, can be obtained by using special values of $\mathbf{M}(\nu, \chi)$.

Proof. The argument is similar to that in the proof of Theorem 3.2 in [MW]. If $\operatorname{Re}\mu > 0$, $\operatorname{Im}\nu \neq 0$ the inner product $I(\nu) = \langle \tilde{\mathbf{M}}(\nu, \phi_r, \chi), f \rangle$ can be computed by using Lemma 2.4 and the fact that C acts by multiplication by $(\nu^2 - 1)/4$ on $H_\infty^{\xi, \nu}$:

$$(4) \qquad \begin{cases} I(\nu) = \int\limits_{\Gamma \backslash G} \sum\limits_{\gamma \in \Gamma_P \backslash \Gamma} \phi(\gamma g) M(\nu, \gamma g, \phi_r) \chi(\gamma^{-1}) \overline{f(g)}\, dg \\ \quad = \int\limits_{\Gamma_N \backslash N} \int\limits_{A} \phi(a) a^{-2\rho} \eta(n) M(\nu, a, \phi_r) \overline{f(na)}\, dn\, da \\ \quad = \dfrac{4\, \overline{c_{\bar\mu}(\eta, f)}}{\nu^2 - \mu^2} \int\limits_{A} \phi(a) \Phi(\nu, \mu)(a)\, da \end{cases}$$

where $a^{2\rho}\Phi(\nu, \mu)(a)$ equals

$$\begin{aligned} &C(w(\nu)(\pi_\nu(a)\phi_r))\, \overline{\mathcal{J}(\bar\mu)(\pi_{\bar\mu}(a)\phi_r)} - w(\nu)(\pi_\nu(a)\phi_r) \overline{C(\mathcal{J}(\bar\mu)(\pi_{\bar\mu}(a)\phi_r))} \\ &= \frac{H^2 - 2H}{4}\, M(\nu, a, \phi_r)\, \overline{\mathcal{J}(\bar\mu)(\pi_{\bar\mu}(a)\phi_r)} \\ &\quad - M(\nu, a, \phi_r)\, \frac{H^2 - 2H}{4}\, \overline{\mathcal{J}(\bar\mu)(\pi_{\bar\mu}(a)\phi_r)}. \end{aligned}$$

In the last equality we have used that

$$XY(w(\nu)(\pi_\nu(a)\phi_r)) \mathcal{J}(\bar\mu)(\pi_{\bar\mu}(a)\phi_r) = w(\nu)(\pi_\nu(a)\phi_r) XY\, \overline{(\mathcal{J}(\bar\mu)(\pi_{\bar\mu}(a)\phi_r))}.$$

Integrating by parts in (4), we get that

$$I(\nu) = -\frac{\overline{c_{\bar\mu}(\eta, f)}}{\nu^2 - \mu^2} \int\limits_{A} H\phi(a) \psi(\nu, \mu)(a)\, da$$

where $\Phi(\nu, \mu)(a) = H\psi(\nu, \mu)(a)$ and $\psi(\nu, \mu)(a)$ equals

$$H(a^{-\rho}M(\nu,a,\phi_r))a^{-\rho}\mathcal{J}(\bar{\mu})(\pi_{\bar{\mu}}(a)\phi_r)$$
$$-a^{-\rho}M(\nu,a,\phi_r)H(a^{-\rho}\overline{\mathcal{J}(\bar{\mu})(\pi_{\bar{\mu}}(a)\phi_r)}).$$

We note that $H\psi(\nu,\pm\nu)(a)=\Phi(\nu,\pm\nu)(a)\equiv 0$, hence it follows that $\psi(\nu,\nu)=K(\nu)$, a holomorphic function independent of a. To determine $K(\nu)$ we may assume that $\mathrm{Re}\,\nu>1$ and let $t\to-\infty$. From Proposition 2.3 we have that $a_t^{-\rho}w(\nu)(\pi_\nu(a_t)\phi_r)\sim a_t^{\nu}/(\Gamma(\nu+1))$ and $a_t^{-\rho}\mathcal{J}(\nu)(\pi_\nu(a_t)\phi_r)\sim a_t^{-\nu}c_r(\nu)$, and this implies that $K(\nu)=(2c_r(\nu))/(\Gamma(\nu))$.
Now, since $H\phi$ has compact support we find

$$(5)\qquad \begin{cases} \lim\limits_{\nu\to\mu}(\nu^2-\mu^2)I(\nu) = -\overline{c_{\bar{\mu}}(\eta,f)}\int\limits_A H\phi(a)\lim_{\nu\to\mu}\psi(\nu,\mu)(a)\,da\\ \qquad = -\overline{c_{\bar{\mu}}(\eta,f)}\int\limits_A H\phi(a)\psi(\mu,\mu)(a)\,da\\ \qquad = \overline{c_{\bar{\mu}}(\eta,f)}\,e^{-ir\pi}\,\dfrac{2c_r(\mu)}{\Gamma(\mu)}.\end{cases}$$

Thus, if $\mu\neq 0$ we get

$$(6)\qquad \begin{cases} \langle \mathrm{Res}_{\nu=\mu}\mathbf{M}^{\eta}(\nu,\phi_r,\chi),f\rangle = \lim\limits_{\nu\to\mu}(\nu-\mu)\langle\tilde{\mathbf{M}}_d(\nu,\phi_r,\chi),f\rangle\\ \qquad = \overline{c(\eta,f)}\;\dfrac{2(4\pi|\lambda|)^{(1-\mu)/2}}{\mu\Gamma\left(\dfrac{1+\mu-sg(\lambda)r}{2}\right)},\end{cases}$$

as asserted. If $\mu=0$ and r is not an odd integer such that $sg(r)=sg(\lambda)$, the same argument yields

$$(7)\qquad \begin{cases} \langle \lim\limits_{\nu\to 0}\nu^2\mathbf{M}^{\eta}(\nu,\phi_r,\chi),f\rangle = \lim\limits_{\nu\to 0}\nu^2\langle\tilde{\mathbf{M}}_d(\nu,\phi_r,\chi),f\rangle\\ \qquad = \overline{c(\eta,f)}\;\dfrac{2(4\pi|\lambda|)^{1/2}}{\Gamma\left(\dfrac{1-sg(\lambda)r}{2}\right)}.\end{cases}$$

Now, if for some f as in the statement $c(\eta,f)\neq 0$, then, if $\mu\neq 0$ (resp. $\mu=0$), it follows from (6) (resp. (7)) that $\mathbf{M}^{\eta}(\nu,\phi_r)$ has a simple (resp. double) pole at $\nu=\mu$. Conversely, if $\mathbf{M}^{\eta}(\nu,\phi_r)$ has a pole at $\mu\neq 0$, then if we take $f=\mathrm{Res}_{\nu=\mu}\mathbf{M}^{\eta}(\nu,\phi_r,\chi)$, (6) implies that $c(\eta,f)\neq 0$. We get to the same conclusion if $\mathbf{M}^{\eta}(\nu,\phi_r)$ has a double pole at $\nu=0$, by using (7).

We now consider the case when $\mu=0$ and r is an odd integer such that $sg(r)=sg(\lambda)$. The argument in the proof of (i) gives in this case

$$(8)\qquad \lim_{\nu\to 0}\langle\nu\tilde{\mathbf{M}}(\nu,\phi_r,\chi),f\rangle = \lim_{\nu\to 0}-\overline{c_0(\eta,f)}\int\limits_A H\phi(a)\frac{\psi(\nu,0)(a)}{\nu}\,da.$$

If, for simplicity, we write $M(\nu,a,\phi_r):=w(\nu)(\pi_\nu(a)\phi_r)$, it follows from Proposition 2.3 and Equation (11) that $\mathcal{J}(0)(\pi_0(a)\phi_r)=b_rM(0,a,\phi_r)$, with $b_r=(4\pi e^{i\pi r/2}/(|r|-1))(-1)^{(|r|-1)/2}$, if $|r|\neq 1$, $b_{\pm 1}=\pm 2\pi i$.

Now we apply the argument used for $r = 1$ in [MW, Proposition 4.5]. The function

$$\psi_1(z,\nu)(a) = H(a^{-\rho}M(z,a,\phi_r))a^{-\rho}\overline{M(\bar{\nu},a,\phi_r)} \\ - a^{-\rho}M(z,a,\phi_r)H(a^{-\rho}\overline{M(\bar{\nu},a,\phi_r)})$$

satisfies $\psi_1(\nu,z) = -\overline{\psi_1(\bar{z},\bar{\nu})}$ and $\psi_1(0,0) = 0$. Now, if $\alpha = \lim_{z\to 0}(\psi_1(z,0))/z$ and $\beta = \lim_{z\to 0}(\psi_1(0,z))/z$, this implies that $\alpha = -\bar{\beta}$. Now $\psi_1(\nu,z) = \alpha\nu + \beta z +$ higher order terms in ν, z, hence it follows that

$$\alpha = \lim_{z\to 0}\frac{\psi_1(z,0)}{z} = \frac{1}{2}\left\{\lim_{z\to 0}\frac{\psi_1(z,z)}{z} + \lim_{z\to 0}\frac{\psi_1(z,-z)}{z}\right\}. \tag{9}$$

Now, as in the case of $\psi(\nu,\pm\nu)$, $\psi_1(z,\pm z)(a)$ does not depend on a, and by using asymptotic information as $t\to -\infty$, one obtains that $\lim_{z\to 0}(\psi_1(z,0))/z = 1$.

Since r is an odd integer, the expression of g_r in (iii) of Theorem 3.2 implies that $\mathrm{Res}_{\nu=0}\,\mathbf{M}(\nu,\phi_r,\chi) = f_r$ is a square integrable automorphic form. Thus, by (8), (9) and using the value of $c_0(\eta,f)$ in Lemma 2.5, we get

$$\begin{aligned}\langle \mathrm{Res}_{\nu=0}\,\mathbf{M}(\nu,\phi_r,\chi),f\rangle &= \langle \mathrm{Res}_{\nu=0}\,\tilde{\mathbf{M}}_d(\nu,\phi_r,\chi),f\rangle \\ &= \lim_{\nu\to 0}\nu\langle \tilde{\mathbf{M}}(\nu,\phi_r,\chi),f\rangle \\ &= \overline{c(\eta,f)}(-1)^{(|r|+1)/2}(4\pi|\lambda|)^{1/2}(1)_{(|r|-1)/2}\end{aligned}$$

as claimed. The last assertion follows from this formula as in case (i). This completes the proof of the theorem. □

4. SPECIAL VALUES OF $\mathbf{M}(\nu,\chi)$

The purpose of this section is to use the family $\mathbf{M}(\nu,\chi)$ in the construction of systems of generators for the spaces of holomorphic automorphic forms of any real weight $r > 0$. We will assume that $\Gamma_N = \{n(m) \mid m \in \mathbf{Z}\}$.

As usual, let $\chi \in \hat{\Gamma}$, $\eta \in \hat{N}$ be such that $\eta_{|\Gamma_N} = \chi_{|\Gamma_N}$, $\chi_{|M} = \chi_\tau$, $\tau \in (-1,1]$. If $\bar{g} = \pi(g)$, we write $g \in G$ uniquely, $g = \sigma(\bar{g})m(g)$, with $m(g) \in M$, where $\sigma : G_0 \mapsto G$ is as in §2.1. We let $v = v_\chi$ be the multiplier system on $\bar{\Gamma} = \pi(\Gamma)$ so that $v_\chi(\bar{\gamma}) = \chi(\sigma(\bar{\gamma}))$, if $\gamma \in \Gamma$.

For $r \equiv \tau\,(2)$, $r > 2$ and $\lambda \in \mathbf{R}^*$, the holomorphic Poincaré series of weight r, parameter λ and multiplier system $v = v_\chi$ is defined by

$$G_r(z,\lambda,v) = \sum_{\bar{\gamma}\in\overline{\Gamma_P}\backslash\bar{\Gamma}} \frac{e^{2\pi i\lambda\bar{\gamma}\cdot z}}{(cz+d)^r}\, v(\bar{\gamma}^{-1}) \tag{1}$$

where $\bar{\gamma} = \left[\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right]$.

We now define $\bar{G}_r(g,\lambda,\chi) := G_r(\bar{g}\cdot i,\lambda,v_\chi)j(\bar{g},i)^{-r}\chi(m(g)) \in C^\infty(\Gamma\backslash G,\chi)[\phi_r]$, where $j(\bar{g},z) = cz+d$ for $\bar{g} = \left[\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right]$. Using the definitions one computes that

$$\bar{G}_r(g,\lambda,\chi) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} e^{2\pi i \lambda \bar{\gamma} g \cdot i} j(\bar{\gamma} g, i)^{-r} \chi(m(\gamma g)) \chi(\gamma^{-1})$$
$$= \sum_{\gamma \in \Gamma_P \backslash \Gamma} \eta_\lambda(n(\gamma g)) a(\gamma g)^{r\rho} e^{-2\pi\lambda a(\gamma g)^{2\rho}} \phi_r(k(\gamma g)) \chi(\gamma^{-1}).$$

We now recall from Proposition 2.3 and (11) in §1 that if $r > 0$, $l \in \mathbf{Z}^{\geq 0}$, $\xi = \phi_{r|M}$ and $a \in A$ then

(2) $$\Gamma(r) M(\xi, r-1, a, \phi_{r+2l}) = (4\pi\lambda)^{-(r/2)} M_{(r+2l)/2,(r-l)/2}(4\pi\lambda a^{2\rho})$$

(3) $$W_{(r+2l)/2,(r-1)/2}(y) = (-1)^l (r)_l M_{(r+2l)/2,(r-1)/2}(y) = y^{r/2} e^{-(y/2)} p_l(y)$$

where $p_l(y)$ is a polynomial of degree l and $p_0(y) = 1$. Also, if $r > 0$ and $y > 0$

$$M_{-(r/2),(r-1)/2}(y) = \varepsilon M_{r/2,(r-1)/2}(-y) = \varepsilon y^{r/2} e^{y/2}$$

where $|\varepsilon| = 1$. Thus we have by (2) and (3)

(4) $$\begin{cases} M^{\eta_\lambda}(\xi, r-1, a, \phi_r, \chi) = \dfrac{a^{r\rho}}{\Gamma(r)} e^{-2\pi\lambda a^{2\rho}}, \\ M^{\eta_\lambda}(\xi, r-1, a, \phi_{-r}, \chi) = \dfrac{\varepsilon a^{r\rho}}{\Gamma(r)} e^{2\pi\lambda a^{2\rho}}. \end{cases}$$

Now, using (2) and (4) we have the following identities (cf. [MW] Lemma 4.1), which show that special values of the **M**-series yield classical Poincaré series (modulo the identification of G_r with $\tilde{G}_r$). If $r > 2$, $\lambda \neq 0$ and $\chi_M = \xi = \phi_{r|M}$, then

(5) $$\mathbf{M}^{\eta_\lambda}(\xi, r-1, g, \phi_r, \chi) = \frac{1}{\Gamma(r)} \tilde{G}_r(g, \lambda, \chi)$$

(6) $$\mathbf{M}^{\overline{\eta_\lambda}}(\bar{\xi}, r-1, g, \phi_{-r}, \bar{\chi}) = \frac{\varepsilon}{\Gamma(r)} \overline{\tilde{G}_r(g, \lambda, \chi)}.$$

In the next result we will show that $\mathbf{M}(\xi, \nu, g, \phi_{r+2l}, \chi)$ is holomorphic at $\nu_r = r-1$ for $1 < r \leq 2$, $l \geq 0$, $\eta = \eta_\lambda$ with $\lambda > 0$ (this is also included in [Br2], Theorem 11.3.9, and in [MW] for $r = 2$, $\chi = 1$). Furthermore, if $l = 0$ this special value corresponds to a holomorphic cusp form, and yields a natural notion of holomorphic Poincaré series for any real weight $r > 1$. Using this notion, we will prove that if $1 < r \leq 2$, these Poincaré series, as $\lambda > 0$ varies, generate the space of (χ, Γ)-holomorphic cusp forms of weight r, and multiplier system $v = v_\chi$.

Theorem 4.1. *Let* $r > 0$, $\nu_r = r-1$, $\lambda \in \mathbf{R}^*$, *be such that* $\chi_{|\Gamma_N} = \eta_{\lambda|\Gamma_N}$. *Set* $\chi_\lambda = \chi^{sg(\lambda)}$. *Then*

(i) *If* $r > 1$, $l \in \mathbf{Z}^{\geq 0}$, $\mathbf{M}^{\eta_\lambda}(\nu, \phi_{sg(\lambda)(r+2l)}, \chi_\lambda)$ *is holomorphic at* $\nu = \nu_r$ *and the value at* $\nu = \nu_r$ *is a cusp form. If* $f \in L^2_d(\Gamma \backslash G, \chi)_K[\phi_{sg(\lambda)(r+2l)}]$ *generates a* $(\mathfrak{g}, K)$*-module isomorphic to* D_r^+, *if* $\lambda > 0$, *or* D_{-r}^-, *if* $\lambda < 0$, *then*

$$(7)\qquad \langle \mathbf{M}^{\eta_\lambda}(\nu_r, \phi_{sg(\lambda)(r+2l)}, \chi_\lambda), f\rangle = c_{\lambda,r,l}\,\overline{c(\eta_\lambda, f)}$$

with $c_{\lambda,r,l} \neq 0$. *Furthermore* $c_{\lambda,r,0} = ((4\pi|\lambda|)^{1-r/2}/2(r-1))$.

The family $\{\mathbf{M}^{\eta_\lambda}(\nu_r, \phi_r, \chi_\lambda) \mid \lambda > 0\}$ (*resp.* $\{\mathbf{M}^{\eta_{-\lambda}}(\nu_r, \phi_{-r}, \chi_{-\lambda}) \mid \lambda > 0\}$), *generates the space of* χ_λ-*holomorphic* (*resp.* $\bar{\chi}_{-\lambda}$-*antiholomorphic*) *cusp forms of weight* r (*resp.* $-r$).

(ii) *If* $0 < r \leq 1$, $l \in \mathbf{Z}^{\geq 0}$, *then* $\mathbf{M}^{\eta_\lambda}(\nu, \phi_{sg(\lambda)(r+2l)}, \chi_\lambda)$ *has at most a simple pole at* $\nu = -\nu_r$. *If* $l = 0$, $\lambda > 0$ (*resp.* $\lambda < 0$), *the residue corresponds to a square integrable,* χ_λ-*holomorphic* (*resp.* $\overline{\chi_\lambda}$-*antiholomorphic*) *automorphic form.*

If $f \in L^2_d(\Gamma\backslash G, \chi)_K[\phi_{sg(\lambda)r}]$ *generates a* $(\mathfrak{g}, K)$-*module isomorphic to* D^+_r, *if* $\lambda > 0$, *or* D^-_{-r} *if* $\lambda < 0$, *then*

$$(8)\qquad \langle \mathrm{Res}_{\nu=-\nu_r}\mathbf{M}^{\eta_\lambda}(\nu, \phi_{sg(\lambda)r}, \chi_\lambda), f\rangle = \frac{2(4\pi|\lambda|)^{r/2}}{\Gamma(2-r)}\, c(\eta_\lambda, f)$$

The family $\{\mathrm{Res}_{\nu=-\nu_r}\mathbf{M}^{\eta_\lambda}(\nu, \phi_r, \chi_\lambda) \mid \lambda > 0\}$ (*resp.* $\{\mathrm{Res}_{\nu=-\nu_r}\mathbf{M}^{\eta_{-\lambda}}(\nu, \phi_{-r}, \chi_\lambda) \mid \lambda > 0\}$) *generates the space of* χ_λ-*holomorphic* (*resp.* $\overline{\chi_\lambda}$-*antiholomorphic*) *square integrable automorphic forms of weight* r (*resp.* $-r$).

Proof. We will first show that if $\lambda > 0$, $r > 1$, $\mathbf{M}^\eta(\nu, \phi_{r+2l}, \chi)$ is holomorphic at $\nu = \nu_r$. Set $f = \mathrm{Res}_{\nu=\nu_r}\mathbf{M}^\eta(\nu, \phi_{r+2l}, \chi)$. By Theorem 3.2, f is a square integrable automorphic form and furthermore, if $\mathrm{Re}\,\nu > 0$, $\mathrm{Im}\,\nu \neq 0$

$$\begin{aligned}\lim_{\nu\to\nu_r}(\nu-\nu_r)\langle\tilde{\mathbf{M}}(\nu, \phi_{r+2l}, \chi), f\rangle &= \lim_{\nu\to\nu_r}(\nu-\nu_r)\langle\tilde{\mathbf{M}}_d(\nu, \phi_{r+2l}, \chi), f\rangle\\ &= \|f\|^2\end{aligned}$$

We now show this limit is zero, hence $f = 0$, as claimed. We have, by (3) and Lemma 2.5 in §1,

$$(9)\qquad \begin{cases} f_{P,\eta}(a) = c(\eta, f)\, W_{(r+2l)/2,(r-1)/2}(4\pi\lambda a^{2\rho})\\ \qquad\quad = (-1)^l (r)_l c(\eta, f) M_{(r+2l)/2,(r-1)/2}(4\pi\lambda a^{2\rho})\\ \qquad\quad = c(\eta, f)(-1)^l \Gamma(r+l)(4\pi\lambda)^{r/2} M(\nu_r, a, \phi_{r+2l})\end{cases}$$

We compute the inner product $\langle\tilde{\mathbf{M}}_T(\nu, \phi_{r+2l}, \chi), f\rangle$ for $\mathrm{Re}\,\nu > 1$ and then extend the validity by analytic continuation.

$$\begin{aligned}&\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma_P\backslash\Gamma}\phi(\gamma g)M(\nu, \gamma g, \phi_{r+2l})\chi(\gamma^{-1})\overline{f(g)}\,da\\ &= \int_{\Gamma_N\backslash N}\int_A \phi(a)\eta(n)M(\nu, a, \phi_{r+2l})\overline{f(na)}\,a^{-2\rho}\,dn\,da\\ &= \overline{c(\eta, f)}\,\Gamma(r+l)(4\pi\lambda)^{r/2}(-1)^l\\ &\quad\times\int_A a^{-2\rho}\phi(a)M(\nu, a, \phi_{r+2l})\overline{M(\nu_r, a, \phi_{r+2l})}\,da\end{aligned}$$

Now the integrand is bounded in absolute value by $\phi(a)a^{\mathrm{Re}\,\nu+\nu_r}$, an integrable function for $\mathrm{Re}\,\nu + \nu_r > 0$, thus $\lim_{\nu\to\nu_r}\langle\tilde{\mathbf{M}}_T(\nu, \phi_{r+2l}, \chi), f\rangle$ equals

$$(10)\qquad \overline{c(\eta,f)}\Gamma(r+l)(4\pi\lambda)^{r/2}(-1)^l\int_A a^{-2\rho}\phi_T(a)|M(\nu_r,a,\phi_{r+2l})|^2\,da$$

hence $\lim_{\nu\to\nu_r}(\nu-\nu_r)\langle\tilde{\mathbf{M}}_T(\nu,\phi_{r+2l},\chi),f\rangle=0$ and thus $\mathbf{M}^\eta(\nu,\phi_{r+2l},\chi)$ is holomorphic at $\nu=\nu_r$, as asserted. Furthermore, it is clear from the argument above that (10) is still valid for any f having the properties in (i).

We now estimate $\psi_T(g)=\mathbf{M}(\nu,g,\phi_{r+2l},\chi)-\tilde{\mathbf{M}}_T(\nu,\phi_{r+2l},g,\chi)$. By the definition of $\tilde{\mathbf{M}}_T(\nu)$, if $\operatorname{Re}\nu>1$ we have

$$(11)\qquad \psi_T(g)=\sum_{\gamma\in\Gamma_P\backslash\Gamma}(1-\phi_T)(\gamma g)M(\nu,\gamma g,\phi_{r+2l})\chi(\gamma^{-1})$$

Since the series above is given locally by a finite sum, it follows by analytic continuation that (11) holds for any $\nu\in\mathbf{C}$.

Now, for any P-Siegel set $\mathcal{S}$, $\operatorname{supp}(\psi_T)-\mathcal{S}$ is compact. On the other hand, $\mathcal{S}=\mathcal{S}_o$ can be chosen so that if $\gamma\in\Gamma$ is such that $\gamma\mathcal{S}_o\cap\mathcal{S}_o\neq\emptyset$, then $\gamma\in\Gamma_P$. Thus, if $a_t\in\mathcal{S}_o$ we have

$$(12)\qquad \psi(a_t)=(1-\phi_T)(a_t)M(\nu_r,a_t,\phi_{r+2l},\chi).$$

Since by (4), this function is rapidly decreasing as $t\to+\infty$, it follows from (12) that $\psi_T(g)$ is bounded. Furthermore, $\tilde{\mathbf{M}}_T(\nu_r,g,\phi_{r+2l},\chi)\to\mathbf{M}(\nu_r,g,\phi_{r+2l},\chi)$ uniformly as $T\to\infty$. These facts imply that $\mathbf{M}(\nu_r,g,\phi_{r+2l},\chi)$ lies in $L^2(\Gamma\backslash G,\chi)$, since $\tilde{\mathbf{M}}(\nu_r,\phi_{r+2l},g,\chi)$ is square integrable by Theorem 2.7 (i).

On the other hand, if $l=0$, $\mathbf{M}(\nu_r,g,\phi_r,\chi)$ is annihilated by E^- (see (4)), hence it corresponds to a holomorphic automorphic form. Since the weight is $r>1$, the Q-constant term on each Q-regular cuspidal parabolic is, in absolute value, a constant multiple of $a_Q(g)^{r\rho}$ which is square integrable on a Q-Siegel set if and only if $r<1$, thus the square integrability of $\mathbf{M}(\nu_r,g,\phi_r,\chi)$ implies that it is a cusp form. Finally, for any $l\in\mathbf{Z}^{\geq 0}$, by using the action of E^+ (cf. §1 (4)) we get that

$$2^l(r)_l\mathbf{M}(\nu_r,g,\phi_{r+2l},\chi)=E^{+l}\mathbf{M}(\nu_r,g,\phi_r,\chi)$$

hence $\mathbf{M}(\nu_r,g,\phi_{r+2l},\chi)$ is also a cusp form, as asserted.
Also, if f is as in (i), letting $T\to\infty$ in (10) we have

$$(13)\qquad \begin{aligned}\langle\mathbf{M}^\eta(\nu_r,\phi_{r+2l},\chi),f\rangle&=\overline{c(\eta,f)}\Gamma(r+l)(4\pi\lambda)^{r/2}(-1)^l\\&\quad\times\int_A a^{-2\rho}|M(\nu_r,a,\phi_{r+2l})|^2\,da\end{aligned}$$

Now setting $c_{\lambda,r,l}$ equal to the coefficient of $c(\eta,f)$ in this expression, we get the asserted inner product formula. Furthermore, if $l=0$ we compute

$$(14)\qquad \begin{aligned}\overline{c(\eta,f)}\,\frac{(4\pi\lambda)^{r/2}}{\Gamma(r)}\int_{-\infty}^{\infty}e^{(r-1)2t}e^{-4\pi\lambda e^{2t}}\,dt&=\overline{c(\eta,f)}\,\frac{(4\pi\lambda)^{1-r/2}}{2\Gamma(r)}\int_0^\infty y^{(r-2)}e^{-y}\,dy\\&=\overline{c(\eta,f)}\,\frac{(4\pi\lambda)^{1-r/2}}{2(r-1)}\end{aligned}$$

Finally, if $l=0$ and $f\neq 0$ is a holomorphic cusp form, then $c(\eta,f)\neq 0$ for some

$\eta = \eta_\lambda$ with $\lambda > 0$, hence (14) is not zero. This concludes the proof of the assertions in (i), in the case of D_r^+.

In the case of a negative weight $\phi_{-(r+2l)}$ we note that if $\mathrm{Re}\,\nu > 1$

$$\begin{aligned}\mathbf{M}^{\bar\eta}(\nu, g, \phi_{-(r+2l)}, \bar\chi) &= \sum_{\gamma\in\Gamma_P\backslash\Gamma} M^{\bar\eta}(\nu, \gamma g, \phi_{-(r+2l)})\bar\chi(\gamma^{-1}) \\ &= \overline{\mathbf{M}^{\eta}(\bar\nu, g, \phi_{r+2l}, \chi)}\end{aligned}$$

hence this identity holds for all ν. It follows that $\mathbf{M}^{\bar\eta}(\nu, \phi_{-(r+2l)}, \bar\chi)$ is holomorphic at $\nu = \nu_r$ and defines a cusp form of weight $-(r+2l)$, which is antiholomorphic if $l = 0$. If $f \in L^2_d(\Gamma\backslash G, \chi)_K[\phi_{-(r+2l)}]$, generates a $(\mathfrak{g}, K)$-module isomorphic to D^-_{-r}. Then $\bar f \in L^2(\Gamma\backslash G, \chi)[\phi_{r+2l}]$ generates a $(\mathfrak{g}, K)$-module isomorphic to D_r^+ and if $\mathrm{Re}\,\nu > 1$ $\langle \tilde{\mathbf{M}}^{\bar\eta}(\nu, \phi_{-r-2l}, \bar\chi), f\rangle = \langle \tilde{\mathbf{M}}^{\eta}(\bar\nu, \phi_{r+2l}, \chi), \bar f\rangle$. By letting $T \to \infty$ we thus get

$$(15) \qquad \langle \mathbf{M}^{\bar\eta}(\nu, \phi_{-r-2l}, \bar\chi), f\rangle = \overline{\langle \mathbf{M}^{\eta}(\bar\nu, \phi_{r+2l}, \chi), \bar f\rangle}$$

Now if we apply (14) and (15), we see that the asserted inner product formula holds also in this case. This concludes the proof of (i).

(ii) Let $0 < r \le 1$. By application of E^- we see that $\mathrm{Res}_{\nu=1-r}\mathbf{M}^\eta(\nu, \phi_r, \chi)$ corresponds to a holomorphic χ-automorphic form of weight r. Furthermore, if f is a holomorphic form of weight r, then f generates a $(\mathfrak{g}, K)$-module isomorphic to D_r^+. If $f \in L^2(\Gamma\backslash G, \chi)$, we may apply Theorem 3.3 and conclude that

$$(16) \qquad \langle \mathrm{Res}_{\nu=1-r}\mathbf{M}^\eta(\nu, \phi_r, \chi), f\rangle = 2^{\delta_{r,1}} \frac{(4\pi\lambda)^{r/2}}{\Gamma(2-r)} \overline{c(\eta, f)}$$

Again, if $f \neq 0$ is a holomorphic automorphic form of weight r, then there exists some $\eta = \eta_\lambda$ with $\lambda > 0$ such that $c(\eta, f) \neq 0$, hence the completeness assertion follows. The assertion in the antiholomorphic case follows similarly from Theorem 3.3. The theorem is now completely proved. $\square$

Corollary 4.2. *Let $\eta = \eta_\lambda$ with $\lambda > 0$, $\eta_{|\Gamma_N} = \chi_{|\Gamma_N}$. We have:*

(i) *If $r > 1$, then $\mathbf{M}^\eta(r-1, g, \phi_r, \chi) = 0$ if and only if for any (χ, Γ)-holomorphic cusp form f of weight r, $c(\eta, f) = 0$.*

(ii) *If $0 < r \le 1$, then $\mathrm{Res}_{\nu=-\nu_r}\mathbf{M}^\eta(\nu, g, \phi_r, \chi) = 0$, if and only if for any (χ, Γ)-holomorphic automorphic form f of weight r, $c(\eta, f) = 0$.*

An entirely analogous result holds for antiholomorphic automorphic forms.

Remark 4.4.

1. For $0 < r < 1$ we note that by the functional equation (19), the residue of $\mathbf{M}(\nu, \phi_r, \chi)$ at $\nu = -\nu_r$ is a multiple of the value, at $\nu = \nu_r < 0$, of the sum of $\mathbf{M}(\nu, \phi_r, \chi)$ and a linear combination of Eisenstein series. Hence one can generate the holomorphic automorphic forms of weight r, either with residues of

$\mathbf{M}^{\eta}(\nu, \phi_r, \chi)$ at $\nu = -\nu_r$, as in (iii), or with values at $\nu = \nu_r$, of $\mathbf{M}^{\eta}(\nu, \phi_r, \chi)$ plus a linear combination of Eisenstein Series.

2. We note that in the case of weight $0 < r < 1$ the residues in the theorem are not in general cusp forms. For instance the theta function $\theta(z)$ is a holomorphic automorphic form of weight $1/2$ for $\Gamma = \Gamma_\theta$, the theta group, with the theta multiplier $v_\theta = v_{\chi\theta}$ and which is not cuspidal (see [Br2], Ch. 14). Thus, Corollary 4.2 implies that for suitable choices of η, if $r = 1/2$, $\chi = \chi_\theta$, the residue $\mathrm{Res}_{\nu=-\nu_r} \mathbf{M}^{\eta}(\nu, g, \phi_r, \chi)$, corresponds to an automorphic form which is not perpendicular to $\theta(z)$, hence it is not a cusp form.

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